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ON TWO NOTIONS OF STRUCTURAL STABILITY

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Introduction

In the literature one can find two notions of structural stability. First the original one given by Andronov and Pontriagin (see [1], [2], [5]) stated for vector fields, that is, for the actions of the additive group of real number $\boldsymbol{R}$ on a manifold M. This definition says roughly that an $\boldsymbol{R}$-action on M is structurally stable if, for any other $\boldsymbol{R}$-action close-to, in the sense that the vector fields generating these actions are close, there exists a homeomorphism of M onto itself mapping the orbits of the first action onto the orbits of the second. This definition can readily be extended (see below § 1), to actions on M of a given real Lie group G in particular $G = Z =$ additive group of all integers.

Another definition was proposed more recently by Smale (see [8] and [9]) for Z-actions on M. Such an action is generated by an diffeomorphism $\phi: M \to M$. Smale's definition is roughly that ϕ is structurally stable if any diffeomorphism ψ sufficiently close to ϕ in the C^1-topology is topologically conjugate to ϕ. Smale's definition, which can also be extended to action on M of any given real Lie group G, seems more restrictive than the one of Andronov and Pontrjagin.

The purpose of this note is to show that in the case $G = Z$, the two definitions are equivalent if the dimension of $M > 1$ and M is connected.

In § 1 we give precise statements of the two definitions, first in the case $G = Z$ (which is the one of interest to us) and then in the general case, for comparison sake.

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1. Definitions of structural stability

A C^1-Z-action on a compact C^∞ manifold M is generated by a C^1-diffeomorphism $\phi: M \to M$.

Definition 1 (*Andronov-Pontrjagin*). A C^1-diffeomorphism $\phi: M \to M$ is structurally stable if for any $\varepsilon > 0$ there exists a neighborhood U of ϕ in

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$\mathrm{Diff}^1(M)$ for the C^1-topology ($\mathrm{Diff}^1(M)$ = group of all C^1-diffeomorphisms of M onto itself) such that for any $\psi \in U$ there exists a homeomorphism $h: M \to M$ with the following two properties: (a) h maps the orbits of ϕ onto the orbits of ψ. (b) $d(x, h(x)) \leq \varepsilon$ for all $x \in M$ where d is some chosen metric on M compatible with its topology. A homeomorphism with property (b) is called an ε-homeomorphism.

Definition 2 (*Smale*). A C^1-diffeomorphism $\phi: M \to M$ is said to be structurally stable if for any $\varepsilon > 0$ there exists a neighborhood U of ϕ in $\mathrm{Diff}^1(M)$ for the C^1-topology such that for any $\psi \in U$ there exist an ε-homeomorphism $h: M \to M$ and t equal to $+1$ or -1 such that $\psi = h \circ \phi^t \circ h^{-1}$.

In fact in Smale's definition $t = 1$ always. We introduce this little complication in Smale's definition in order to square it off with the general definition. It will follow from what will be proved later that $t = 1$ always (provided one chooses U small enough).

Now we give the same definitions in the general case of any Lie group G. A C^1-action of G on a manifold M is a homomorphism $\phi: G \to \mathrm{Diff}^1(M)$ (also called representation) such that the mapping $G \times M \to M: (g, x) \to \phi(g)[x]$ is of class C^1. Call $A^1(G, M)$ the set of all these C^1-actions. It is a subset of $C^0(G, \mathrm{Diff}^1(M))$, the set of all continuous mappings $G \to \mathrm{Diff}^1(M)$. This set carries the compact open topology ($\mathrm{Diff}^1(M)$ being endowed with the C^1-topology). Hence $A^1(G, M)$ as a subset of $C^0(G, \mathrm{Diff}^1(M))$ is endowed, by restriction, of the compact open topology. The orbit of $x \in M$ under the G-action ϕ is the set $\{\phi(g)(x) \mid g \in G\}$.

Definition 1. A C^1-action $\phi_0: G \to \mathrm{Diff}^1(M)$ of G on M is said to be structurally stable if for any $\varepsilon > 0$ there exists a neighborhood U of ϕ_0 in $A^1(G, M)$ with the compact open topology such that for any $\phi \in U$ there exists an ε-homeomorphism $h: M \to M$ mapping the orbits of ϕ_0 onto the orbits of ϕ.

Definition 2. A C^1-action $\phi_0: G \to \mathrm{Diff}^1(M)$ of G on M is said to be structurally stable if for any $\varepsilon > 0$ there exists a neighborhood U of ϕ_0 in $A^1(G, M)$ with the compact open topology such that for any $\phi \in U$ there exist an ε-homeomorphism $h: M \to M$ and an algebraic automorphism $t: G \to G$ such that $h \circ \phi(g) = \phi_0(t(g)) \circ h$ for all $g \in G$.

It is obvious that Definitions 1 and 2 stated previously in the case $G = Z$ are particular cases of the general Definitions 1 and 2 stated above.

Remark (*due to J. Palis*). For $G \neq Z$ it is obvious that in general Definition 1 does not imply Definition 2. In fact in the case $G = \boldsymbol{R}$ consider a Morse-Smale vector field with 2 or more closed orbits with different periods. It seems that it would be more appropriate to have the reparametrization t to depend on the points on the manifold too so that in Definition 2 one should replace $t: G \to G$ by $t: G \times M \to M$ and the following equation by:

$$h(\phi(g)x) = \phi_0(t(g, x))h(x) .$$

Now we state our main result.

Theorem. *In the case $G = Z$ the two definitions are equivalent at least if* $\dim M \geq 1$ *and M is connected.*

It is obvious that Definition 2 implies Definition 1. So we only have to show the converse. This will follow from some lemmas.

Comment on the case $\dim M = 1$. In that case $M = S^1$. Our proof below does not cover that case but the theorem is true in that case. The proof (due to J. Palis) is as follows: by Peixotos theorem [5] the structurally stable, in the sense of Definition 1, Z-actions are the Morse-Smale actions, and then it is an easy particular case of a theorem of J. Palis [3] that the Morse-Smale actions are stable in the sense of Definition 2.

2. Some auxiliary lemmas

Lemma 1. (a) *If $\phi: M \to M$ and $\psi: M \to M$ are two C^1-diffeomorphisms such that there exists a homeomorphism $h: M \to M$ mapping the orbits of ϕ onto the orbits of ψ, then h maps the set* $\mathrm{Per}(\phi)$ *of all periodic points of ϕ onto the set* $\mathrm{Per}(\phi)$ *of all periodic points of ψ.*

(b) *If a C^1-diffeomorphism $\phi: M \to M$ is structurally stable, then* $\mathrm{Per}(\phi)$ *is countable.*

Proof. All this is well known. We only give the proofs for the sake of completeness. (a) follows immediately from the fact that a point $X \in M$ is periodic for ϕ if and only if its orbit is compact, and compactness is preserved by a homeomorphism. To prove (b) we use a general approximation theorem (see [4] or [7] or [8]) which implies that the set E of all C^1-diffeomorphisms ϕ, such that $\mathrm{Per}(\psi)$ is countable, is a Baire subset of $\mathrm{Diff}^1(M)$. Hence choose a $\psi \in E$ so close to ϕ that there exists a homeomorphism $h: M \to M$ mapping the orbits of ϕ onto the orbits of ψ. Then by (a) h maps $\mathrm{Per}(\phi)$ onto $\mathrm{Per}(\psi)$, and hence $\mathrm{Per}(\phi)$ is countable.

Lemma 2. *Assume that* $\dim M > 1$ *and M is connected, and that D is a countable subset of M. Then $M - D$ is arcwise connected.*

Proof. Let $U_1 \cup U_2 \cup \cdots \cup U_n$ be a finite covering of M, U_j, $j = 1, \cdots, n$, being domains of charts (U_i, α_i), and α_i mapping U_i onto the unit open ball $B^d(0, 1)$ in the euclidean spare R^d ($d =$ dimension of M). Let $D_j = D \cap U_j$ and $C_j = \alpha_j(D_j)$. It is obviously sufficient to prove that for x and y in $U_j - D_j$ there exists a continuous curve in $U_j - D_j$ joining x to y or what is the same that there exists a continuous curve in $B^d(0, 1) - C_j$ joining $\alpha_j(x)$ to $\alpha_j(y)$. Let E be the 2-plane in R^d spanned by $0, \alpha_j(x), \alpha_j(y)$ (if $\alpha_j(x) = 0$ or $\alpha_j(y) = 0$, take any 2-plane containing 0 and $\alpha_i(y)$ or $\alpha_j(x)$). Let $\Delta(x)$ be the set of all lines in E joining $\alpha_j(x)$ to the points of $C_j \cap E$, and $\Delta(y)$ the corresponding set for $\alpha_j(y)$. $\Delta(x)$ and $\Delta(y)$ are countable. Hence they exist, as close as we want to the line $(\alpha_j(x), \alpha_j(y))$, a line $\delta x \notin \Delta(x)$ passing through $\alpha_j(x)$ and a line $\delta_y \notin \Delta(y)$ passing through $\alpha_j(y)$. These lines meet at a point ζ close to the segment $\overline{\alpha_j(x)\alpha_j(y)}$. As this segment is contained in $B^d(0, 1)$ so will be ζ; as $B^d(0, 1)$ is convex, the 2 segments $\overline{\alpha_j(x)\zeta}$ and $\overline{\zeta\alpha_j(y)}$ will be in $B^d(0, 1)$ and, in fact, in $B^d(0, 1) - C_j$ by construction since $\overline{\alpha_j(x)\zeta}$ lies on δ_x and $\overline{\zeta\alpha_j(y)}$ on δ_y. Hence the polygonal curve $\overline{\alpha_j(x)\zeta} \cup \overline{\zeta\alpha_j(y)}$ joins $\alpha_j(x)$ to $\alpha_j(y)$ in $B^d(0, 1) - C_j$, and Lemma 2 is proved.

3. Proof of the theorem

Assume ϕ and ψ are two C^1-diffeomorphisms $M \to M$ such that there exists a homeomorphism $h: M \to M$ mapping the orbits of ϕ onto the orbits of ψ and that $\operatorname{Per}(\phi)$ is countable (hence by Lemma 1 (a) $\operatorname{Per}(\psi)$ is). We are going to show that there exists an integer $q \in Z$ such that $\psi^q \circ h = h \circ \phi$.

Choose a point x_0 in $M - \operatorname{Per}(\phi)$. Then there exists an unique integer $q \in Z$ such that $\psi^q(h(x_0)) = h(\phi(x_0))$. So we are going to show that if y is any other point in $M - \operatorname{Per}(\phi)$, then $\psi^q(h(y)) = h(\phi(y))$. By Lemma 2 there exists a continuous arc $\gamma \subset M - \operatorname{Per}(\phi)$, joining x_0 to y parametrized by the continuous map $t \in I = [0, 1] \to x(t) \in M - \operatorname{Per}(\phi)$, $x(0) = x_0$, $x(1) = y$. Let $T_m = \{t \mid t \in I, \psi^m(h(x(t))) = h(\phi(x(t)))\}$.

Lemma 3. (a) *The T_m are closed.* (b) $T_m \cap T_k = \emptyset$ *if* $m \neq k$.
(c) $\bigcup_{m \in Z} T_m = I$.

Proof. (a) follows from the fact that the 2 functions $t \to \psi^m(h(x(t)))$ and $t \to h(\phi(x(t)))$ are continuous.

(b) If $t_0 \in T_m \cap T_k$, then $\psi^m(h(x(t_0))) = h(\phi(x(t_0))) = \psi^k(h(x(t_0)))$. Hence $\psi^{m-k}(h(x(t_0))) = h(x(t_0))$. So $h(x(t_0)) \in \operatorname{Per}(\psi)$. By Lemma 1 (a) $x(t_0) \in \operatorname{Per}(\phi)$. But $x(t_0) \in \gamma \subset M - \operatorname{Per}(\phi)$, a contradiction.

(c) is obvious for given any $x \in M$, $h(\phi(x)) \in \psi$-orbit of $h(x)$ and hence there exists an n such that $\psi^n(h(x)) = h(\phi(x))$. If $x \in M - \operatorname{Per}(\phi)$, this n is unique; otherwise *not*.

Let J_m be the interior of T_m, and let $\omega = \bigcup_{m \in Z} J_m$.

Lemma 4. (a) *ω is open and dense in I.*

(b) *Every connected component of ω is contained in one and only one T_m.*

Proof. (a) Since I is the union of the countable class of closed sets T_m, (a) follows from the Baire property of I.

(b) Since ω is the union of the open *disjoint* sets J_m, the components of ω are those of the J_m. Hence (b) follows from this and Lemma 3 (b).

Let $K = I - \omega$. If K is empty, $\omega = I$. I is the sole connected component of ω and hence contained in a unique T_m by Lemma 4 (b). As $0 \in T_q$ ($x(0) = x_0$ and $\psi^q(h(x_0)) = \phi(h(x_0))$ by assumption), $m = q$ so $1 \in T_q$ and $\psi^q(h(y)) = \psi^q(h(x(1))) = h(\phi(y))$. We will show that $K \neq \emptyset$ leads to a contradiction. Let $K_m = K \cap T_m$. Then the K_m are closed and $K = \bigcup_{m \in Z} K_m$. As a closed subset of I, K has the Baire property, so one of the set K_m, say K_p, has a nonempty interior in the space K. This means that there exists an open interval $\delta \subset I$ such that $\delta \cap K \neq \emptyset$ and $\delta \cap K \subset K_p$. By Lemma 4 (a), $\delta \cap \omega \neq \emptyset$. Let

$]a, b[= \{t \mid a < t < b\}$ be a connected component of ω meeting δ. Then one of the extremities a, b is contained in δ, for otherwise $]a, b[\supset \delta$ and then $\delta \cap K = \emptyset$. Assume for example that $a \in \delta$. As $a \in K$, $a \in \delta \cap K \subset K_p$. So $a \in T_p$. But by Lemma 4 (b), $]a, b[$ is contained in a unique T_k. As $a \in$ closure of $]a, b[$ and this closure is also contained in T_k (T_k being closed), $a \in T_k$. As $T_k \cap T_p = \emptyset$ if $k \neq p$, by Lemma 3 (b) it follows that $k = p$. *So we have proved that if a connected component of ω meets δ, then it is contained in T_p.* Thus $\delta \cap \omega \subset T_p$. As $\delta \cap K \subset K_p \subset T_p$, it follows that $\delta \subset T_p$; hence $\delta \subset J_p$ as δ is open. But then $\delta \subset \omega$ which contradicts the fact that $\delta \cap K \neq \emptyset$. So $K = \emptyset$, and we have proved the following:

Lemma 5. $\psi^q \circ h = h \circ \phi$.

Lemma 6. $q = \pm 1$.

Proof. If $q \neq \pm 1$, then $h_0\phi^k = \psi^{qk} \circ h$ for all $k \in Z$, (easy to see by induction), and hence $h(\phi\text{-orbit of } x) = \{\psi^{qk}(h(x)) \mid k \in Z\}$. This last set is not the whole orbit of x, unless $q = \pm 1$ if $x \notin \operatorname{Per}(\phi)$.

Finally we have proved the following:

Proposition. *If $\phi, \psi \colon M \to M$ are two C^1-diffeomorphisms such that there exists a homeomorphism $h \colon M \to M$ mapping the orbits of ϕ onto the orbits of ψ, then $\psi^q \circ h = h \circ \phi$ where $q = 1$ or -1.*

Now assume $\phi \colon M \to M$ is structurally stable in the sense of Definition 1. For any $\varepsilon > 0$ there exists a neighborhood U of ϕ in $\operatorname{Diff}^1(M)$ such that for any $\psi \in U$ there exists an ε-homeomorphism $h \colon M \to M$ mapping the orbits of ϕ onto the orbits of ψ. By the proposition it follows that $\psi^q \circ h = h \circ \phi$ where $q = +1$ or -1. But there exists a possibly smaller neighborhood $U' \subset U$ of ϕ in $\operatorname{Diff}^1(M)$ such that if $\psi \in U'$ then $q = 1$. For, if such a neighborhood did not exist, then we could find a sequence $\{\psi_j \mid j = 1, 2, \cdots\}$, $\psi_j \in U$ and $\psi_j \to \phi$ in $\operatorname{Diff}^1(M)$ as $j \to +\infty$, and a sequence $\{h_j \mid j = 1, 2, \cdots\}$ of homeomorphisms such that h_j maps the orbits of ϕ onto the orbits of ψ and $\sup_{x \in M} (x, h_j(x)) \to 0$ as $j \to \infty$, and $\psi_j^q \circ h_j = h_j \circ \phi$ with $q = -1$. Thus taking the limits as $j \to +\infty$ we get $\phi^{-1} = \phi$, and all points in M would be periodic of period 2 contradicting Lemma 1 (a). Hence the theorem is proved.

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